MATH2050C Assignment 9

Deadline: March 25 , 2025.

Hand in: 5.1 no. 4ac, 5; 5.2 no. 1ac, 12, 13, 15; Suppl Problems no. 2, 3.

Section 5.1 no. 3, 4ac, 5, 8, 13.

Section 5.2 no. 1bc, 4, 7, 12, 13, 15.

Supplementary Problems

1. Determine the largest domain on which the function is defined and study its continuity.
   (a) $\sin x/x$.
   (b) $\sqrt{\dfrac{x+6}{x+1}}$.
   (c) $\operatorname{sgn}(x^2 - x - 2)$.
   (d) $e^{1/\sin x}$.
2. Show that the function
$$f(x) = \frac{(1+x)^{1/2} - 1}{(1+x)^{1/3} - 1}$$
   can be extended to be continuous at $x = 0$.
3. Let f be defined in A. Suppose f is continuous at $c \in A$ and $f(c) > 0$. Show that there is some $\delta > 0$ such that $f(x) > 0$ for $x \in A, |x - c| < \delta$.

See next page

Continuity of Functions

A real-valued function is a function from a subset of $\mathbb{R}$ to $\mathbb{R}$. It consists of two components, namely, its domain of definition and its rule of assignment. For instance, define $F(n)$ on $\mathbb{N}$ by the rule $F(n)$ is the number of prime factors in n. Here the domain of definition is specified to be $\mathbb{N}$. For instance, $F(8) = 1$ since $8 = 2^3$ has one prime factor, and $F(54) = 2$ since $54 = 2 \times 3^3$. Sometimes, only a formula is given as the rule of assignment, its domain of definition is understood to be the largest set on which the formula makes sense. For instance, the formula $\sin x/x$ defines a function on $\mathbb{R} \setminus \{0\}$, and $(x^2-4)/(x+2)$ defines a function on $\mathbb{R} \setminus \{-2\}$.

Let f be a function defined on A, a nonempty subset of $\mathbb{R}$ and $c \in A$. We call f to be continuous at c if for every $\varepsilon > 0$, there is some $\delta > 0$ such that $|f(x) - f(c)| < \varepsilon$ for all $x \in A, |x - c| < \delta$. Or, according to the Sequential Criterion, whenever $x_n \to c, x_n \in A, f(x_n) \to f(c)$.

For $c \in A$, either c is a cluster point of A or it is an isolated point of A, that is, $(c-\delta_0, c+\delta_0) \bigcap A = \{c\}$ for some δ_0. When c is a cluster point, f is continuous at c iff $\lim_{x \to c} f(x) = L$ exists and $L = f(c)$. On the other hand, f is always continuous at an isolated point c.

Using results from Limits of Functions, one shows that continuity is preserved under all algebraic operations. Furthermore, by Sequential Criterion, one shows that it is also preserved under compositions of functions.

Here are some examples. We call a function continuous on $B \subset A$ if it is continuous at every point in B.

- All polynomials are continuous on $\mathbb{R}$.
- All rational functions $p(x)/q(x)$ are well-defined and continuous on $\{x : q(x) \neq 0\}$.
- The exponential function $E(x)$ and $\sin x, \cos x$ are continuous on $\mathbb{R}$ (see previous exercises).
- The function $E(x)\sqrt{(1 + \tan^2 x)/(2 - |\cos x|)}$ is well-defined and continuous on $\{x : x \neq n\pi + \pi/2, n \in \mathbb{Z}\}$.

Here are some examples of discontinuous points.

- The function $sgn(x) = x/|x|, x \neq 0$, and $sgn(0) = 0$ has a jump discontinuity at $c = 0$.
- The function $f(x) = \sin 1/x, x \neq 0$ and $\sin 0 = 0$ is well-defined on $[0, \infty)$ and it is not continuous at $c = 0$ (due to rapid oscillation).
- The function $h(x) = 1/x, x \neq 0$ and $h(0) = 0$ is well-defined on $\mathbb{R}$ and it is not continuous at $c = 0$ (it becomes unbounded near 0).
- The function $g(x) = e^{1/x}, x \neq 0$ and $g(0) = 0$ is well-defined on $\mathbb{R}$ and it is not continuous at $c = 0$ ($\lim_{x \to 0^-} g(x) = 0$ and $\lim_{x \to 0^+} g(x) = \infty$.)

You need to know the Dirichlet's function which is discontinuous everywhere and the Thomae's function which is continuous at all irrationals but discontinuous at all rationals.